Valuing Constant-Growth Perpetuities with Finance Intuition (Instead of Infinite Series)

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The formula for calculating the present value of a finite level annuity is one of the simplest equations in corporate finance. It is PV = PMT/r, where PV is the present value, PMT is the payment, and r is the periodic interest rate. However, corporate finance instructors and textbooks rarely discuss the derivation of this equation. This is because the typical derivation requires a knowledge of calculus beyond that of most business majors. In this paper, I demonstrate that it is possible to use basic ideas from finance – how competitive markets set the price of an asset equal to the asset's market value – at the same place in the derivation that usually requires taking limits.

INTRODUCTION

One of the most common time value of money and dividend discount model formulas is the constant growth formula. Typically, instructors present this formula as a "black box," with little or no explanation of how it is derived. Even when the presenter derives the formula, the derivation, typically taking the limit of a geometric series, does not build on the students' financial intuition. (One notable exception is Berk, DeMarzo, and Harford (2016)).

In this paper, I present a derivation of the constant growth formula by creating a loan that has geometrically increasing payments, where the borrower does not completely pay off the loan principal each period. I amortize this loan and show for a lender that is willing to make the loan, i.e., where the present value equals the loan principal, it is possible to solve for the present value of the loan. By looking in more depth at how to construct a loan with constantly increasing payments, students can gain a deeper understanding of constant growth perpetuities.

LEVEL PERPETUITIES

Perhaps one of the most surprising results in time value of money is that it is easier to calculate the present value of an infinite stream of level payments (or an infinite stream of payments that grow at a constant rate) than it is to calculate the present value of a finite stream of two or more payments.

Every finance instructor has probably taught these formulas:

PV = PMT / r	Where PV = present value,
	PMT = payment,
	r = the required rate or return,
$PV_0 = PMT1 / (r - g)$	$g = the \ constant \ growth \ rate.$

There is a lot of financial content in these formulas. For students who have studied the convergence of geometric series, the typical derivation of these formulae is mathematically rigorous and straightforward. However, 1) most business undergraduates are not comfortable with infinite series, and 2) such a derivation typically does not use any financial intuition.

Another way to approach this problem is to begin with something more familiar. For example, an instructor could amortize a typical loan and ask the following questions:

1) What happens to the number of payments when your client pays a smaller payment?

Response: The number of payments increases.

2) Why?

Response: The borrower repays less principal each period.

3) What happens to the loan when all your client does is pay the interest every year?

Response: The loan principal remains the same every year. It becomes an "interest-only" loan with PMT = r * BPB, where BPB = "beginning principal balance" = present value.

If prices are set competitively, so that the market price of the infinite level stream of cash payments equals the lump sum payment at time zero, the analyst can use algebra to manipulate the equation PMT = r * BPB. This is the step where an appeal is made to financial intuition rather than taking a formal, mathematically rigorous approach using limits. Divide both sides by r to derive the formula for the present value of a perpetuity. Now, in addition to having the formula for the PV of a level perpetuity, students understand the relationship between an interest-only loan and the present value of a perpetuity.

To illustrate, suppose that a potential borrower can make payments of \$5,000 per year for the foreseeable future. The interest rate is 5%. How much would the bank be willing to lend, assuming that prices are set competitively (and the only fees are included in the interest rate)? Start with the formula for the calculation of one "Interest-only" payment, then solve for BPB.

 $PMT = r * BPB \rightarrow BPB = PMT/r$ BPB = \$5,000/0.05 = \$100,000

Since prices are set competitively (and the only fees are in the interest payment), the beginning principal balance is equal to the present value of the loan. Thus,

PV = BPB = \$5,000

The following three figures illustrate how to present this information in a finance class. Figure 1 displays an example for a loan of \$100,000, an annual payment of \$40,000 every year except the last, and an annual interest rate of 10%.

Figure 1: Finite Annuity Amortization Schedule, \$40,000 Annual Payment.

	А	В	С	D	E	F				
1	Loan amount =		\$100,000	USD						
2	Payment =		\$40,000	USD	* •	4000				
3	Interest Rate	e =	10.0%		* •	100				
4	Note: The final payment will be less than what is listed and will									
5	pay off the lo	oan.								
6										
	Beginning	ginning				Ending				
	Principal	Interest	Principal	Additional	Total	Principal				
7	Balance	Payment	Payment	Borrowing	Payment	Balance				
8	\$100,000	\$10,000	\$30,000		\$40,000	\$70,000				
9	\$70,000	\$7,000	\$33,000		\$40,000	\$37,000				
10	\$37,000	\$3,700	\$36,300		\$40,000	\$700				
11	\$700	\$70	\$700		\$770	\$0				

Figure 2 displays the result for a loan of \$100,000, payments of \$20,000, and an interest rate of 10%.

Figure 2: Finite Annuity Amortization Schedule, \$20,000 Annual Payment.

	A B		С	D	E	F	
1	Loan amount =		\$100,000	USD			
2	Payment =		\$20,000	USD	-	2000	
3	Interest Rate	e =	10.0%		* *	100	
4	Note: The fir	nal paymer	nt will be le	ss than wha	t is listed a	nd will	
5	pay off the lo	ban.					
6							
	Beginning					Ending	
	Principal Interest		Principal	Additional	Total	Principal	
7	Balance	Payment	Payment	Borrowing	Payment	Balance	
8	\$100,000	\$10,000	\$10,000		\$20,000	\$90,000	
9	\$90,000	\$9,000	\$11,000		\$20,000	\$79,000	
10	\$79,000	\$7,900	\$12,100		\$20,000	\$66,900	
11	\$66,900	\$6,690	\$13,310		\$20,000	\$53,590	
12	\$53,590	\$5,359	\$14,641		\$20,000	\$38,949	
13	\$38,949	\$3,895	\$16,105		\$20,000	\$22,844	
14	\$22,844	\$2,284	\$17,716		\$20,000	\$5,128	
15	\$5,128 \$513		\$5,128		\$5,641	\$0	

Figure 3 displays the first three payments for an interestonly loan of \$100,000, payments of \$10,000, and an interest rate of 10%.

Figure 3: Interest-only Amortization Schedule, \$10,000 Annual Payment.

	A B		С	D	E	F	
1	Loan amoun	t =	\$100,000	USD			
2	Payment =		\$10,000	USD	* *	1000	
3	Interest Rate	e =	10.0%		* *	100	
4	Note: The fir	nal paymen	it will be le	ss than wha	t is listed a	nd will	
5	pay off the lo	ban.					
6							
	Beginning	eginning				Ending	
	Principal	Interest	Principal	Additional	Total	Principal	
7	Balance	Payment	Payment	Borrowing	Payment	Balance	
8	\$100,000	\$10,000	\$0		\$10,000	\$100,000	
9	\$100,000	\$10,000	\$0		\$10,000	\$100,000	
10	\$100,000	\$10,000	\$0		\$10,000	\$100,000	

As shown in Figure 3, if the borrower never pays any principal, the principal balance remains the same, so the interest payment also remains the same.

CONSTANT GROWTH PERPETUITIES

The instructor can extend the results of the level perpetuity to the constant growth case by considering the following example. Now suppose that your client takes out an interest-only loan, but then fails to pay part of the interest payment (g * outstanding loan balance) every year. Assume that r > g > 0 (so that some interest is paid, but not all of it). Also, the analyst can appeal to financial intuition by assuming that the value of the two sides of the transaction (the lump sum at time zero and the stream of incomplete interest payments) have the same value. This allows the analyst to treat the beginning principal balance as the value of the loan. This justifies the use of algebra to solve for the PV of the growth perpetuity.

The total payment made in this case is (r - g) * BPB.

The interest payment is r * BPB.

Note that (r - g) * BPB < r * BPB

What happens to the loan in this case, when your client pays less than the net interest owed at the end of the period? Look at the first four payments of a loan of \$100,000, interest rate 10%, value of g = 2%, resulting in payments of \$8,000 per year (shown below in figure 4).

Figure 4: Growing Annuity Amortization Schedule.

	А	В	С	D	Е	F				
1	A loan where not all the interest is paid each year									
3	Loan amou	nt =	\$100,000	USD						
5	interest rate	e (r) =	10.0%		* *	100				
6	"growth" ra	te (g) =	2.0%		* *	20				
7	Note: The t	otal payme	ents are no	t level in th	is case.					
8										
	Beginning					Ending				
	Principal	Interest	Principal		Total	Principal				
9	Balance	Payment	Payment	"Growth"	Payment	Balance				
10	\$100,000	\$10,000	\$0	\$2,000	\$8,000	\$102,000				
11	\$102,000	\$10,200	\$0	\$2,040	\$8,160	\$104,040				
12	\$104,040	\$10,404	\$0	\$2,081	\$8,323	\$106,121				
13	\$106,121	\$10,612	\$0	\$2,122	\$8,490	\$108,243				

In this case, every year, part of the interest is not paid. So, at the end of the year, the borrower owes more than the beginning principal balance for that year. The bank adds the unpaid interest to the principal balance. As a result, the ending principal balance is larger than the beginning principal balance for any given year. Since the principal balance is growing, and the interest payment is a fixed proportion of the beginning principal balance, the interest payment grows every year. For any year N, calculate the total payment as follows:

Total payment = Beginning principal balance *(r - g), or more concisely,

PMTN = PVN-1 * (r - g)

Solve for PV to get the constant growth formula, PVN-1 = PMTN / (r - g)

ARITHMETIC GROWTH PERPETUITIES

The instructor can extend the results of the level perpetuity to the arithmetic growth case by considering the following example.

Suppose that your client needs a stream of cash flows that grows arithmetically. At the end of year 1, your client will need \$100. At the end of year 2, your client will need an additional \$200. At the end of N years, your client will need an additional N * \$100. The interest rate is 10%. To value this perpetuity, first note that by rearranging the payments, as in Figure 5, payment N can be thought of as N * \$100 by looking at the columns of the table. Next look at the rows of the table. When looking at the rows of Figure 5, this perpetuity appears to be the sum of a stream of constant level perpetuities. This is different from the constant-growth perpetuity examined earlier because the increase in the size of the payments is determined by adding the same value, \$100, to the amount of the previous year's payment. This is illustrated in figure 5.

Figure 5: Arithmetically Increasing Stream of Payments.

	А	В	С	D	E	F	G	Н	- 1
1									
2	interest rate	10%							
3	PMT1	100							
4									
5	CF stream#	Time	1	2	3	4	5	6	
6	1		100	100	100	100	100	100	
7	2			100	100	100	100	100	
8	3				100	100	100	100	
9	4					100	100	100	
10	5						100	100	
11	6							100	
12									
13									
14	Net CF		100	200	300	400	500	600	

Cash flow stream #1 is just a level perpetuity with the first cash flow at the end of the first period. The PV of cash flow stream #1 is equal to

(\$100/0.1000) *(1 + 0.1000) ^-1

Cash flow stream #2 is just like cash flow stream #1 except that the first payment is missing. It is as if every payment in cash flow stream #1 was delayed by one period. This "delayed" perpetuity can be valued by multiplying the value of cash flow stream #1 by (1 + 0.1000) ^-1. The PV of cash flow stream #2 is therefore equal to

(\$100/0.1000) *(1 + 0.1000) ^-2

Cash flow stream #N is just like cash flow stream #1 except that the first N - 1 payments are missing. The PV of cash flow stream N is equal to

(\$100/0.1000) *(1 + 0.1000) ^-N

The analyst may appeal to financial intuition twice to solve this problem. Using the arguments from the level annuity part of the article, it is possible to value each stream of \$100 payments in the table. To begin, each stream of \$100 payments can be valued using the formula

PV = PMT/r = \$100/0.10 = \$1,000.

This simplifies the problem by converting it into a level perpetuity with an annual payment of \$1,000. Next, this stream of \$1,000 level payments can be valued again by using the same arguments as in the previous step to obtain

PV = PMT/r = \$1,000/0.10 = \$10,000.

Note that because the stream of payments is infinite, it would usually not be possible to rearrange the terms in the series without taking limits. However, since it is assumed that markets are competitive, with competitively set prices, it is possible to use algebra to rearrange the terms in the series.

CONCLUSIONS

The surprisingly simple equation for calculating the present value of an infinite stream of level payments typically requires taking the limit of the partial sums of a geometric series. By using the idea from finance of competitively set market prices, it is possible to derive a number of time value of money present value equations and to increase business students' understanding of foundational time value of money principles at the same time.

REFERENCES

Berk, Jonathan, DeMarzo,Peter M., and Harford, Jarrad V. T. 2016. Fundamentals of corporate finance. 4th edition. New York, NY : Prentice Hall, [2016]

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